THE CAUCHY–POISSON PROBLEM FOR A POOL WITH A VERTICAL BARRIER

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The problem of the scattering of a monochromatic wave incident on a submerged vertical plane was initially solved in [1]. Subsequent generalizations of this problem are found in [2-5]. The nonsteady-state problem is dealt with in [6], its solution yielding the asymptote of the flow over long periods of time, but it does not allow tracing the evolution of the process.

The solution of the nonsteady-state problem is written out in the present paper in quadratures, thus making it possible in detail to study its properties. This is achieved by means of the integral of the problem [6] and the method of analytical extension.

1. Let us examine the plane linear initial boundary-value problem

$$\begin{gathered}\Delta\varphi = 0 \text{ in } \Omega,\ \varphi_{tt} + \varphi_y = 0\ (y = 0),\\ \varphi_x = 0 \text{ on } \Gamma,\ \varphi = 0,\ \varphi_t = -\delta(x - x_0)\ \ (t = 0,\ y = 0),\end{gathered} \tag{1.1}$$

which describes the motion of a liquid, generated by the initial perturbation of the free boundary. At the instant of time t = 0 the surface of the liquid exhibits a concentrated elevation of area equal to unity, and in the vicinity of the point x_0, when t > 0, this elevation disintegrates under the action of the forces of gravity. The right-hand Cartesian coordinate system is oriented so that the barrier Γ lies in the x = 0 plane, while the direction of the y axis is opposite to that of the acceleration of free fall. Relationships (1.1) have been written in dimensionless variables, with the length and velocities chosen so that the Froude number of the problem and the depth of barrier submersion are equal to unity. The potential of the velocities $\varphi(x, y, t, x_0)$ depends on x_0 as well as on the parameter, the flow region $\Omega = \{x, y | y < 0, x \in R^1\}\backslash\Gamma$, $\Gamma = \{x, y | x = 0, y < -1\}$. The function φ is a fundamental solution of the Cauchy–Poisson problem in the Ω region, since with the aid of the latter the solution of the general problem is written out in quadratures.

We have to determine a solution to problem (1.1) such as to satisfy the following additional conditions:

$$\begin{gathered}|\nabla\varphi| \leqslant C[x^2 + (y+1)^2]^{-1/4}\ \ (x, y) \in \Omega,\\ |\nabla\varphi| \in L_2(\Omega),\ |\varphi_t(x, 0, t, x_0)| < +\infty\ (t > 0).\end{gathered} \tag{1.2}$$

The solution for φ can be represented in the form of the sum of the even and odd components of the variable x. The even component of φ_r corresponds to the initial conditions $\varphi_r = 0$, $\varphi_{r_t} = -(\delta(x - x_0) + \delta(x + x_0))/2$ (t = 0, y = 0) and is written explicitly, since for it we have $\partial\varphi_r/\partial x = 0$ when x = 0, y ≤ 0. The odd component of φ_n represents the solution of problem (1.1), in which the initial conditions have to be replaced by the following:

$$\varphi_n = 0,\quad \varphi_{n_t} = -(\delta(x - x_0) - \delta(x + x_0))/2\quad (t = 0,\ y = 0). \tag{1.3}$$

Here it is enough to construct φ_n in the region x > 0, y < 0, since for x = 0, $\varphi_n = 0$ in the case of −1 < y < 0 (the potential is continuous in the flow region) and $\partial\varphi_n/\partial x = 0$ for the case in which y < −1. Subsequently $\varphi_r(x, y, t, x_0)$ is regarded as a known function and we have to determine only the odd component of the potential (the subscript n is dropped where this results in no misunderstanding).

2. We will denote the expression $\varphi_{tt} + \varphi_y$ (x > 0, y < 0, t > 0) in terms of $g(x, y, t, x_0)$. Proceeding formally, it is not difficult to obtain the boundary-value problem for the determination of this new function. From (1.1) we have

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$$\Delta g = 0 \text{ in } \Omega, \quad g = 0 \ (y = 0), \ g_x = 0 \text{ on } \Gamma, \tag{2.1}$$

The additional conditions from (1.2) yield

$$|g| < C_1[x^2 + (y+1)^2]^{-1/4} \ (x \geqslant 0, \ y < 0). \tag{2.2}$$

Let us note that t and x_0 are contained in problem (2.1), (2.2) as parameters. Let $g(x, y, t, x_0)$ be a known function, with the potential φ then found as a solution for the evolution problem

$$\begin{gathered} \varphi_{tt} + \varphi_y = g(x, y, t, x_0) \ (x \geqslant 0, \ y < 0, \ t > 0), \\ \varphi = 0, \ \varphi_t = \varphi_1(x, y, x_0) \ \ (t = 0), \end{gathered} \tag{2.3}$$

in which x, x_0 function in the role of parameters, while $\varphi_1(x, y, x_0)$ is a harmonic function in the case of $x > 0$, $y < 0$ such that $\varphi_{1x} = 0$ in the case of $x = 0$, $y < -1$, $\varphi_1 = 0$ when $x = 0$, $-1 < y < 0$ and $\varphi_1 = -(\delta(x - x_0) - \delta(x + x_0))/2$ for $y = 0$, $x > 0$. The latter condition follows from (1.3).

The function $g(x, y, t, x_0)$ is infinitely differentiable when $x > 0$, $y < 0$, so that therefore $\varphi(x, y, t, x_0)$, determined from (2.3), is a harmonic function for the case in which $x > 0$, $y < 0$, satisfying the condition of nonpenetration to Γ. Indeed, having differentiated Eqs. (2.3) with respect to x and causing x to converge on zero ($y < -1$), we derive a uniform problem for the equation of heat conduction relative to $\varphi_x(0, -y, t, x_0)$. In the class of slow-growth functions such a problem exhibits only a trivial solution. Following the procedures of [6], it is sufficient to solve (2.3) only when $x = 0$. When $x > 0$ the solution for $\varphi(x, y, t, x_0)$ is expressed in terms of $\varphi(0, y, t, x_0) \equiv \theta(t, -y)$ in quadratures.

The solution of problem (2.1) with the additional condition (2.2) is constructed by means of the theory of analytical functions and is determined with an accuracy to some arbitrary factor $a(t, x_0)$:

$$g(x, y, t, x_0) = a(t, x_0) \ \operatorname{Im} \{1/\sqrt{z^2+1}\}, \ z = x + iy,$$

when $x = +0$ $g(+0, y, t, x_0) = -a(t, x_0)/\sqrt{y^2 - 1}$ ($y < -1$).

3. When $x = +0$ in the new variables $\alpha = t$, $\tau = -y$, problem (2.3) assumes the form

$$\begin{gathered} \theta_\tau - \theta_{\alpha\alpha} = a(\alpha, x_0)/\sqrt{\tau^2 - 1} \ (\tau > 1, \ \ \alpha > 0), \\ \theta = 0, \ \theta_\alpha = \varphi_1(0, -\tau, x_0) \ \ (\tau > 1, \alpha = 0) \end{gathered} \tag{3.1}$$

and corresponds to the one-dimensional problem for the nonuniform equation of heat conduction, without any initial conditions, and here, at the boundary of the region ($\alpha = 0$) the value of the sought function ($\theta = 0$) and its first derivative $\theta_\alpha(0, \tau)$ are given. We have to determine both the function $\theta(\alpha, \tau)$ and $a(\alpha, x_0)$ under the additional condition

$$|\theta(\alpha, \tau)| \leqslant C_2(\tau)\alpha^k \ (\alpha > 0, \ \tau > 1) \tag{3.2}$$

(k is a positive quantity independent of τ). Condition (3.2) corresponds to the slow-growth condition for $\varphi_t'(0, y, t, x_0)$ as $t \to \infty$.

The function $a(\alpha, x_0)$ describes the distribution along the shaft ($\alpha > 0$) of heat sources and their behavior when $\tau > 1$ is known. For example, the shaft may include impurities whose particles under specific conditions "explode" with evolution of heat. We are familiar in this case with the occurrence of an "explosion" of a single particle; however, the distribution of the "exploding" particles along the shaft is not given and must be found from the known flow of heat $\theta_\alpha(0, \tau)$ through the end of the shaft.

The Laplace transform for τ, such as that used in (3.1), where $\theta(\alpha, \tau)$, the right-hand side of the equation and the initial conditions of extension to zero in the region $\tau < 1$, $\alpha > 0$, leads to an ordinary differential equation whose general solution is as follows:

$$\theta^L(\alpha, p) = C_1(p, x_0)\, e^{\sqrt{p}\alpha} + C_2(p, x_0)\, e^{-\sqrt{p}\alpha} + \\ + \frac{K_0(p)}{2\sqrt{p}} \int_0^\infty a(\alpha_0, x_0)\, e^{-\sqrt{p}|\alpha - \alpha_0|}\, d\alpha_0$$

$\left(\theta^L(\alpha, p) = \int_0^\infty \theta(\alpha, \tau)\, e^{-\tau p}\, d\tau, \operatorname{Re} p \geqslant 0, K_0(p)\right.$ is the MacDonald function of zeroth order, $K_0(p) = \left.\int_1^\infty e^{-\tau p}(\tau^2 - 1)^{-1/2}\, d\tau\right)$. The limitation imposed in (3.2) yields $C_1 \equiv 0$ [7], while the conditions when $\alpha = 0$ lead to the following equations:

$$C_2(p, x_0) = -\theta_\alpha^L(0, p)/2\sqrt{p}, \quad \int_0^\infty a(\alpha_0, x_0)\, e^{-\sqrt{p}\alpha_0}\, d\alpha_0 = = \theta_\alpha^L(0, p)/K_0(p), \tag{3.3}$$

The second of which serves for the determination of the function $a(\alpha, x_0)$ a [$K_0(p)$ has no roots when $-\pi < \arg p \leq \pi$ [8]]. If it is found, then $\theta(\alpha, \tau)$ is written in quadratures

$$\theta(\alpha, \tau) = \frac{1}{2\sqrt{\pi}} \int_0^\infty a(\alpha_0, x_0)\, k(\alpha - \alpha_0, \tau)\, d\alpha_0 - \frac{1}{2\sqrt{\pi}} \int_1^\tau e^{-\frac{\alpha^2}{4(\tau - \tau_0)}} \frac{\varphi_1(0, -\tau_0, x_0)}{\sqrt{\tau - \tau_0}}\, d\tau_0, \tag{3.4}$$
$$k(\alpha, \tau) = \int_1^\tau e^{-\frac{\alpha^2}{4(\tau - \tau_0)}} \frac{d\tau_0}{\sqrt{\tau - \tau_0}\sqrt{\tau_0^2 - 1}}.$$

In (3.3) we will denote the right-hand side in terms of $F(p)$, $\operatorname{Re} p > 0$. The function $F(p)$ permits an analytical extension into the left-hand half-plane $\operatorname{Re} p < 0$ with a branch along the negative portion of the real axis. We will denote the values of $F(p)$ at the lower and upper edges of this branch as $F^-(\sigma)$ and $F^+(\sigma)$ ($p = \sigma e^{\pm i\pi}$), respectively. Then, for the complex variable $s = \sqrt{p}$ ($\sqrt{+1} = 1$) Eq. (3.3) yields the Laplace transform of $a(\alpha, x_0)$ for the case in which $\operatorname{Re} s \geq 0$. The function $F(s^2)$ is analytical and exhibits no singular points for $\operatorname{Re} s > 0$, so that $a(\alpha, x_0)$ is therefore determined by the Bromwich integral

$$a(\alpha, x_0) = \frac{1}{2\pi i} \int_{c - i\infty}^{c + i\infty} e^{s\alpha} F(s^2)\, ds \quad (c > 0),$$

which as $c \to +0$, $s = i\mu$, $\mu \in R^1$ yields

$$a(\alpha, x_0) = \frac{1}{2\pi} \left(\int_{-\infty}^0 F^-(\mu^2)\, e^{i\mu\alpha}\, d\mu + \int_0^\infty F^+(\mu^2)\, e^{i\mu\alpha}\, d\mu \right). \tag{3.5}$$

Formulas (3.4) and (3.5) uniquely define the solution of problem (3.1) under the additional conditions of (3.2). As proof of this assertion it is sufficient to establish, first of all, that $F(p)$ permits the analytical extension, without any singular points, to the entire plane with the branch along the ray $\operatorname{Re} p < 0$, $\operatorname{Im} p = 0$; and secondly, that $F(p) \to 0$ as $|p| \to \infty$, $|\arg p| < \pi$; thirdly, the integrals $\int_0^\infty F^\pm(\sigma)\sigma^{-1/2}\, d\sigma$ converge absolutely.

4. Dropping the cumbersome calculations, we present the expression

$$\theta_\alpha(0, \tau) = -\frac{x_0}{\pi} \sqrt{\frac{\tau^2 - 1}{x_0^2 + 1}} \frac{1}{\tau^2 + x_0^2},$$

with the Laplace transform of this function

$$\theta_\alpha^L(0, p) = -\frac{x_0}{\pi\sqrt{x_0^2 + 1}} \left[K_0(p) - (x_0^2 + 1) \int_1^\infty \frac{e^{-\tau p}\, d\tau}{\sqrt{\tau^2 - 1}\,(\tau^2 + x_0^2)} \right].$$

The analytical extension of the MacDonald function $K_0(p)$ to the left-hand half-space $\operatorname{Re} p < 0$ with the branch along the negative portions of the real axis is given by the following formula from [8]:

$$K_0(p) = K_0(-p) \pm \pi i I_0(-p) \; (\mathrm{Re}\, p < 0, \; \mathrm{Im}\, p \lessgtr 0),$$

where $I_0(p)$ is a modified Bessel function, $I_0(-ip) = J_0(p)$, $I_0(-p) = I_0(p)$. At the edges of the branch (we identify the upper edge of the branch with a plus sign, while the lower edge is identified with a minus sign)

$$K^+_0(x) = K_0(-x) - \pi i I_0(x), \; K_0^-(x) = \overline{K_0^+(x)}, \; x < 0. \tag{4.1}$$

We are left only with the need to construct the analytical extension to the region $\mathrm{Re}\, p < 0$ of the integral

$$J(p) = \int_1^\infty \frac{e^{-\tau p}\, d\tau}{\sqrt{\tau^2 - 1}\,(\tau^2 + x_0^2)}, \tag{4.2}$$

after which the function $F(p)$ will be determined for the entire plane with a branch along the ray $\mathrm{Re}\, p < 0$, $\mathrm{Im}\, p = 0$. Using

$$(\tau^2 + x_0^2)^{-1} = |x_0|^{-1} \int_0^\infty e^{-|x_0|\eta} \cos \tau\eta \, d\eta,$$

we rewrite (4.2) as follows:

$$J(p) = \frac{1}{2|x_0|} \int_0^\infty e^{-|x_0|\eta} [K_0(p - i\eta) + K_0(p + i\eta)]\, d\eta = \frac{1}{2|x_0|} (J_1(p) + J_2(p)).$$

It is not difficult to see that $J_2(p) = \overline{J_1(\bar{p})}$, $J_1(p) = \overline{J_2(\bar{p})}$, so that therefore it is enough to construct the analytical extension of the functions $J_1(p)$, $J_2(p)$ into the region $\mathrm{Re}\, p < 0$, $\mathrm{Im}\, p > 0$ and to use the indicated formulas. In particular, the limit values of $J(p)$ at the upper $[J^+(x)]$ and the lower $[J^-(x)]$ edges of the branch are associated by the relationship $J^+(x) = \overline{J^-(x)}$. With imaginary values for the argument $p = i\lambda$ $(\lambda > 0)$ from (4.1) we have the equations

$$J_1(i\lambda) = J_2(-i\lambda) - \pi i\lambda e^{i|x_0|i\lambda} \int_0^1 e^{-i|x_0|\beta i\lambda} I_0(i\lambda\beta)\, d\beta + \frac{\pi i}{\sqrt{x_0^2 + 1}} e^{i|x_0|i\lambda},$$

$$J_2(i\lambda) = J_1(-i\lambda) + \pi i\lambda e^{-i|x_0|i\lambda} \int_0^1 e^{i|x_0|\beta i\lambda} I_0(i\lambda\beta)\, d\beta - \frac{\pi i}{\sqrt{x_0^2 + 1}} e^{-i|x_0|i\lambda},$$

which will then lead to the formulas for the analytical extension

$$J_1(p) = J_2(-p) - \pi p e^{i|x_0|p} \int_0^1 e^{-i|x_0|\beta p} I_0(p\beta)\, d\beta + \frac{\pi i}{\sqrt{x_0^2 + 1}} e^{i|x_0|p},$$

$$J_2(p) = J_1(-p) + \pi p e^{-i|x_0|p} \int_0^1 e^{i|x_0|\beta p} I_0(p\beta)\, d\beta - \frac{\pi i}{\sqrt{x_0^2 + 1}} e^{-i|x_0|p}$$

$$(\mathrm{Re}\, p < 0, \quad \mathrm{Im}\, p > 0).$$

From this we have

$$J(p) = \frac{1}{2|x_0|} \left\{ 2|x_0| J(-p) - \frac{2\pi}{\sqrt{x_0^2 + 1}} \sin |x_0| p - 2\pi i p \int_0^1 I_0(p\beta) \sin [|x_0| p (1 - \beta)]\, d\beta \right\}$$

when $\mathrm{Re}\, p < 0$, $\mathrm{Im}\, p > 0$. At the upper edge of the branch ($\mathrm{Im}\, p = +0$)

$$J^+(x) = J(-x) - \frac{\pi}{\sqrt{x_0^2 + 1}} \frac{\sin x_0 x}{x_0} - \frac{\pi i}{x_0} \int_0^{|x|} I_0(\beta) \sin (x_0 [|x| - \beta])\, d\beta. \tag{4.3}$$

With consideration of (4.1) and (4.3) the value of F(p) at the upper edge of the branch is given by the expression

$$F^+(\sigma)=-\frac{x_0}{\pi\sqrt{x_0^2+1}}+\frac{x_0}{\pi}\sqrt{x_0^2+1}\,J^+(-\sigma)/K_0^+(-\sigma)\quad(\sigma>0). \tag{4.4}$$

It is clear that at the lower edge $F^-(\sigma)=\overline{F^+(\sigma)}$. We will find the asymptote as $\sigma\to\infty$ of the function $F^+(\sigma)$, determined in (4.4). As $\sigma\to\infty$, since the modified Bessel functions have the asymptotes [8] $I_0(\sigma)=(2\pi\sigma)^{-1/2}e^{\sigma}[1+O(1/\sigma)]$, $K_0(\sigma)=(2\sigma/\pi)^{-1/2}e^{-\sigma}[1+O(1/\sigma)]$, then

$$F^+(\sigma)=-\frac{x_0}{\pi\sqrt{x_0^2+1}}+\frac{1}{\pi}\sqrt{x_0^2+1}\,I_0^{-1}(\sigma)\int_0^{\sigma}I_0(\beta)\sin x_0(\sigma-\beta)\,d\beta+O(e^{-\sigma}).$$

We will represent the convolution of $D_0(\sigma)=\int_0^{\sigma}I_0(\beta)\sin x_0(\sigma-\beta)\,d\beta$ in equivalent form

$$D_0(\sigma)=e^{\sigma}\int_0^{\sigma}e^{-\beta}I_0(\beta)\,e^{-(\sigma-\beta)}\sin x_0(\sigma-\beta)\,d\beta.$$

Here the function $e^{-\rho}\sin x_0\rho$ is absolutely integrable in [0, T], T > 0 and diminishes more rapidly than any power of ρ^{-1} as $\rho\to\infty$, with the function $e^{-\beta}I_0(\beta)$ limited and integrable in [0, T], as $\beta\to\infty$

$$e^{-\beta}I_0(\beta)=\frac{1}{\sqrt{2\pi\beta}}+\frac{1}{8\beta\sqrt{2\pi\beta}}+\ldots$$

Consequently, we can use the following result from [9]:

$$D_0(\sigma)=e^{\sigma}\left\{\frac{1}{\sqrt{2\pi\sigma}}\int_0^{\infty}e^{-\rho}\sin x_0\rho\,d\rho+\frac{1}{2\sigma\sqrt{2\pi\sigma}}\int_0^{\infty}\left(\rho+\frac{1}{4}\right)e^{-\rho}\sin x_0\rho\,d\rho+O(\sigma^{-5/2})\right\}.$$

Finally, we find

$$D_0(\sigma)=\frac{x_0}{x_0^2+1}\frac{e^{\sigma}}{\sqrt{2\pi\sigma}}+\frac{x_0(x_0^2+9)}{8(x_0^2+1)^2}\frac{e^{\sigma}}{\sigma\sqrt{2\pi\sigma}}+\ldots\quad(\sigma\to\infty),$$

from which

$$F^+(\sigma)=\frac{1}{\pi}\frac{x_0}{(x_0^2+1)^{3/2}}\frac{1}{\sigma}+O(\sigma^{-2}). \tag{4.5}$$

Relationship (4.5) shows that formulas (3.4) and (3.5) indeed determine the single solution of problem (3.1), (3.2). Substituting (3.5)-(4.4) into (3.4), we obtain a solution for this problem in the form

$$\theta(\alpha,\tau)=\frac{1}{2\pi^{3/2}}\frac{x_0}{\sqrt{x_0^2+1}}\int_1^{\tau}\frac{\sqrt{\tau_0^2-1}}{\tau_0^2+x_0^2}e^{-\frac{\alpha^2}{4(\tau-\tau_0)}}\frac{d\tau_0}{\sqrt{\tau-\tau_0}}+$$
$$+\frac{1}{2\pi^{3/2}}\int_0^{\infty}\operatorname{Re}F^+(\mu^2)\,k_c(\alpha,\tau,\mu)\,d\mu-\frac{1}{2\pi^{3/2}}\int_0^{\infty}\operatorname{Im}F^+(\mu^2)\,k_s(\alpha,\tau,\mu)\,d\mu,$$

where $k_c(\alpha,\tau,\mu)=\int_0^{\infty}k(\alpha-\alpha_0,\tau)\cos\alpha_0\mu\,d\alpha_0;\quad k_s(\alpha,\tau,\mu)=\int_0^{\infty}k(\alpha-\alpha_0,\tau)\sin\alpha_0\mu\,d\alpha_0;$

$$\operatorname{Re}F^+(\sigma)=-\frac{x_0}{\pi\sqrt{x_0^2+1}}+\frac{x_0}{\pi}\sqrt{x_0^2+1}\,\{l_1(\sigma,x_0)K_0(\sigma)$$
$$+\pi l_2(\sigma,x_0)I_0(\sigma)\}/l_0(\sigma);$$

$$\operatorname{Im}F^+(\sigma)=-\frac{x_0}{\pi}\sqrt{x_0^2+1}\,\{l_2(\sigma,x_0)K_0(\sigma)-\pi l_1(\sigma,x_0)I_0(\sigma)\}/l_0(\sigma);$$

$$l_1(\sigma,x_0)=\int_1^{\infty}\frac{e^{-\tau\sigma}\,d\tau}{\sqrt{\tau^2-1}\,(\tau^2+x_0^2)}+\frac{\pi}{\sqrt{x_0^2+1}}\frac{\sin x_0\sigma}{x_0};$$

$$l_2(\sigma, x_0) = \frac{\pi}{x_0}\int_0^\sigma I_0(\beta)\sin x_0(\sigma-\beta)\,d\beta;\quad l_0(\sigma) = K_0^2(\sigma) + \pi^2 I_0^2(\sigma).$$

Let us note that the functions $k_c(\alpha, \tau, \mu)$, $k_s(\alpha, \tau, \mu)$, $\ell_0(\sigma)$ are independent of the parameter x_0 and the asymptote $\theta(\alpha, \tau)$ for large and small values of $|x_0|$ is therefore determined in terms of the corresponding uniform asymptotes of the functions $\ell_1(\sigma, x_0)$, $\ell_2(\sigma, x_0)$. Thus, as $x_0 \to 0$ it is not difficult to obtain

$$l_1(\sigma, x_0) = \int_1^\infty \frac{e^{-\tau\sigma}\,d\tau}{\tau^2\sqrt{\tau^2-1}} + \pi\frac{\sin x_0\sigma}{x_0} + O(x_0^2)\quad (\sigma > 0),$$

$$l_2(\sigma, x_0) = \pi\sigma^2 I_0(\sigma) + \frac{\pi^2}{2}\sigma\,[I_0(\sigma)L_1(\sigma) - I_1(\sigma)L_0(\sigma)] - \pi\sigma I_1(\sigma) + O(x_0^2)\quad (0 < \sigma < |x_0|^{-1}).$$

Here $L_1(\sigma)$, $L_0(\sigma)$ are modified Struve functions; $L_\nu(\sigma) = e^{-(\nu+1)\pi i/2}H_\nu(e^{\pi i/2}\sigma)$. For $|x_0| \gg 1$

$$l_1(\sigma, x_0) = x_0^{-2}(K_0(\sigma) + \pi\sin x_0\sigma) + O(x_0^{-4}),$$
$$l_2(\sigma, x_0) = \pi x_0^{-2}(I_0(\sigma) - \cos x_0\sigma) + O(x_0^{-3}),$$

in this case the formulas for $\operatorname{Re}F^+(\sigma)$, $\operatorname{Im}F^+(\sigma)$ are simplified:

$$\operatorname{Re}F^+(\sigma) = l_0^{-1}(\sigma)\{K_0(\sigma)\sin x_0\sigma - \pi I_0(\sigma)\cos x_0\sigma\}\operatorname{sgn} x_0,$$
$$\operatorname{Im}F^+(\sigma) = l_0^{-1}(\sigma)\{K_0(\sigma)\cos x_0\sigma + \pi I_0(\sigma)\sin x_0\sigma\}\operatorname{sgn} x_0.$$

5. By means of the constructed function $\theta(\alpha, \tau)$ the solution of the original problem (1.1) is written in quadratures

$$\varphi(x, y, t, x_0) = -\frac{1}{\pi}\int_0^\infty \xi^{-1/2}e^{\xi y}\sin\sqrt{\xi}\,t\cos\xi(x-x_0)\,d\xi -$$
$$-\frac{2}{\pi}\int_0^t\int_1^\infty \theta(t_0, \tau)N(\tau-y, x, t-t_0)\,d\tau\,dt_0 +$$
$$+\frac{4}{\pi}xy\int_1^\infty \tau\theta(t, \tau)\frac{d\tau}{(x^2+(y-\tau)^2)(x^2+(y+\tau)^2)},$$
$$N(\beta, x, \lambda) = \int_0^\infty \xi^{1/2}e^{-\xi\beta}\sin\sqrt{\xi}\,\lambda\sin\xi x\,d\xi.$$

Here the first term yields the solution for the problem in the absence of a barrier [10], and the subsequent terms describe the correction factor for this solution, due to the presence of a submerged plate. The deformation of the free boundary $y = \eta(x, t)$ $[\eta = -\varphi_t(x, 0, t, x_0)]$ is calculated from the formulas

$$\eta(x, t, x_0) = \sum_{n=0}^{3} H_n(x_0)\eta_n(x, t, x_0) + \eta_4^+(x, t, x_0) + \eta_4^-(x, t, x_0) + \eta_5^+(x, t, x_0) + \eta_5^-(x, t, x_0), \tag{5.1}$$

where $\eta_0(x, t) = \frac{1}{\pi}\int_0^\infty \cos\sqrt{\xi}\,t\cos\xi(x-x_0)\,d\xi;\quad H_0(x_0) \equiv 1;$

$$\eta_1(x, t) = \int_0^\infty K_0(\xi)\sin\xi x\left[\sqrt{2}\cos\left(t\sqrt{\xi} - \frac{\pi}{4}\right) - e^{-t\sqrt{\xi}}\right]d\xi;$$
$$H_1(x_0) = \frac{x_0}{2\pi^2\sqrt{x_0^2+1}};$$

$$\eta_2(x, t, x_0) = \int_0^\infty l_3(\xi, x_0) \sin \xi x \left[\sqrt{2} \cos\left(t\sqrt{\xi} - \frac{\pi}{4}\right) - e^{-t\sqrt{\xi}}\right] d\xi;$$

$$H_2(x_0) = -\frac{x_0}{2\pi^2} \sqrt{x_0^2 + 1};$$

$$\eta_3(x, t, x_0) = \int_0^\infty l_6(\xi, x_0) K_0(\xi) \sin \xi x \left[\sqrt{2} \cos\left(t\sqrt{\xi} - \frac{\pi}{4}\right) - e^{-t\sqrt{\xi}}\right] d\xi;$$

$$H_3(x_0) = \pi^{-2};$$

$$\eta_4^{\pm}(x, t, x_0) = \mp \pi^{-2} \int_0^\infty \xi^{1/2} K_0(\xi)\, l_4^{\pm}(\xi, t, x_0) \sin \xi x \, d\xi;$$

$$\eta_5^{\pm}(x, t, x_0) = -\pi^{-2} \operatorname{Im} \int_0^\infty \xi K_0(\xi)\, l_5^{\pm}(\xi, x_0)\, e^{\mp i\sqrt{\xi} t} \sin \xi x \, d\xi;$$

$$l_3(\xi, x_0) = \int_1^\infty \frac{e^{-\tau\xi}\, d\tau}{\sqrt{\tau^2 - 1}\,(\tau^2 + x_0^2)};$$

$$l_4^{\pm}(\xi, t, x_0) = \operatorname{Im} \left\{ \text{V.p.} \int_0^\infty \mu F^+(\mu^2) \frac{e^{i\mu t}\, d\mu}{(\mu \pm \sqrt{\xi})(\mu^2 + \xi)} \right\};$$

$$l_5^{\pm}(\xi, x_0) = \text{V.p.} \int_0^\infty F^+(\mu^2) \frac{d\mu}{(\mu^2 + \xi)(\mu \pm \sqrt{\xi})}; \quad l_6(\xi, x_0) = \operatorname{Im} \int_0^\infty F^+(\mu^2) \frac{d\mu}{i\sqrt{\xi} - \mu}.$$

Bearing in mind the study of the limit regimes of liquid motion above the vertical plate, let us examine the more general problem

$$\begin{gathered} \Delta' U = 0 \ \text{ in } \ \Omega_a, \quad U_{t't'} + \mathrm{Fr}^{-1} U_{y'} = 0 \quad (y' = 0), \\ U_{x'} = 0 \ \text{ on } \ \Gamma_a, \quad U = 0, \quad U_{t'} = -A\delta(x' - x_1) \quad (t' = 0,\ y' = 0), \end{gathered} \tag{5.2}$$

in which $\Gamma_a = \{x', y' \mid x' = 0, y' < -a\}$; $\Omega_a = \{x', y' \mid y' < 0\} \backslash \Gamma_a$; The Froude number $\mathrm{Fr} = V_*^2/gL_*$; L_*, V_* denote, respectively, the characteristic length and the speed of the process.

Let the deformation of the free boundary in (5.2) be described by the equation $y' = h(x', t', x_1)$, in which case it is not difficult to obtain the formula $h(x', t', x_1) = a^{-1} \mathrm{Fr}\, \eta(x'/a, t'/\sqrt{\mathrm{Fr}\, a}, x_1/a)$ from which it follows that as $a \to \infty$

$$h(x', t', x_1) = \pi^{-1} \mathrm{Fr} \int_0^\infty \cos\left(\sqrt{\sigma/\mathrm{Fr}}\, t'\right) \cos \sigma (x' - x_1)\, d\sigma + O(a^{-2}),$$

i.e., the correction factor due to the presence of the vertical plate exhibits an order of $O(a^{-2})$ when $a \gg 1$. When the barrier approaches the free surface ($a \to 0$), then with $x' = O(a)$, $t' = O(\sqrt{a})$, $x_1 = O(a)$ the constructed solution (5.1) loses force: $h = O(a^{-1})$, $a \to 0$, and it becomes necessary to make use of the "internal" asymptotic expansion in the vicinity of the coordinate origin, treating a as the small parameter.

LITERATURE CITED

1. W. R. Dean, "On the reflection of surface waves by a submerged plane barrier," Proc. Cambridge Phil. Soc., 41, No. 2 (1945).
2. F. Ursell, "The effect of a vertical barrier on surface waves in deep water," Proc. Cambridge Phil. Soc., 43, No. 3 (1947).
3. F. Ursell, "On the waves due to the rolling of a ship," Q. J. Mech. Appl. Math., 1, No. 2 (1948).
4. M. D. Khaskind, "Radiation and diffraction of surface waves from a vertical floating plate," PMM, 23, No. 3 (1959).

5. M. Lewin, "The effects of vertical barriers on progressing waves," J. Math. Phys., 42, No. 3 (1963).
6. C. C. Mei, "Radiation and scattering of transient gravity waves by vertical plates," Q. J. Mech. Appl. Math., 19, Pt. 4 (1966).
7. G. E. Shilov, Mathematical Analysis. Second Special Course [in Russian], Nauka, Moscow (1965).
8. G. N. Watson, Theory of Bessel Functions, Cambridge University Press, New York (1972).
9. É. Ya. Riekstyn'sh, "The asymptotic representation of certain types of convolution integrals," Latv. Mat. Ezhegodnik, Latv. Gos. Univ., No. 8 (1970).
10. L. N. Sretenskii, The Theory of Liquid Wave Motions [in Russian], Nauka, Moscow (1977).

THE INVERSE PROBLEM OF STREAMLINING SINGULARITIES WITH THE PLANE FLOW OF AN IDEAL FLUID WITH A FREE BOUNDARY

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In the present study we will examine the plane potential steady-state flow of a heavy ideal fluid with a free boundary. In this case, the velocity potential has a finite number of point singularities. The potential motions of the fluid in the presence of a finite system of point singularities were examined in [1] in connection with the problem of the streamlining of a wing-shaped object under water, and the calculation of the wave resistance, and these were dealt with also in [2, 3], etc. In the cited references the problem was solved in "direct" formulation, i.e., on the basis of given point singularities of a complex potential, which simulated the streamlined solid, and the profile of the free surface and the velocity field were determined. The solutions were derived within the framework of linear wave theory. In the present paper we solve the "inverse" problem: the stationary profile of the free surface is given, and we have to reconstruct the pattern of the flow through the thickness of the fluid. The solution of the problem is achieved both in approximate linear theory and in a precise formulation.

1. Construction of the Solution. Let $S(x)$ $(-\infty < x < \infty)$ represent the profile of the free surface and let v_0 be the velocity of the unperturbed flow. By means of $G \subset C$ we will denote the region occupied by the fluid: $G = \{z : z = x + iy,\ y < S(x)\}$ (we will investigate an infinitely deep fluid). The set of points lying at the surface is denoted S: $S = \{z: \operatorname{Im} z = S(\operatorname{Re} z)\}$. We will impose the following limitation on $S(x)$:

$$S(x) < v_0^2/2g \tag{1.1}$$

(g is the gravitational acceleration). Fulfillment of inequality (1.1) ensures the absence of critical points of complex velocity at the boundary of the fluid and this, in turn, guarantees the smoothness of the profile for $S(x)$ [4]. Let P represent the total number of poles for the complex velocity $w(z)$ in G, i.e., $P = \sum_{i:\, z_i \in G} p_i$, where $P < \infty$ and p_i represent the multiplicity of the pole z_i. The quantities P and p_i are not known in advance and are determined in the process of the solution. Moreover, the natural condition of limitation with respect to the velocity of the fluid at infinity is assumed to be satisfied:

$$|w(z)| < w < +\infty,\ |z| > R,\ z \in G \tag{1.2}$$

(R is a rather large number). The boundary conditions for the potential W are satisfied at the free surface:

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